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GUIDING CENTER HAMILTONIAN THEORY OF FREE-ELECTRON LASERS

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ABSTRACT

The relativistic guiding center ponderomotive Hamiltonian for free-electron lasers is derived. The derivation takes into account arbitrary signal wave polarization and wiggler field geometry including guide field nonuniformities. In particular, it is allowed (i) for a tapered axial guide quasistatic magnetic field along its direction of propagation, (ii) for a realizable wiggler to be tapered both in amplitude and in period, and (iii) for the signal electromagnetic wave to be a growing modulated wave of arbitrary geometry propagating in the direction of the static magnetic field. The equations of motion are then derived including the guiding center perpendicular drifts and beam quasistatic self-fields.

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I. INTRODUCTION

In a previous paper[1] (henceforth referred as "I"), a general expression for the relativistic ponderomotive Hamiltonian of two interacting electromagnetic waves was derived. It was found that relativistic effects introduce new terms in the expression for the ponderomotive Hamiltonian similar to those found earlier[2] in the case of a single electromagnetic wave. The derivation in "I" was as general as possible. It was allowed for arbitrary $k\rho$, $E_\parallel/E_\perp$, ω/Ω, $k_\perp/k_\parallel$, $v/c < 1$, polarization, and for slowly growing and spatially modulated waves. It was necessary to use Hamiltonian theory in order to account for all that and to introduce nonuniformities in the background fields in a systematic way. The calculation was an application of relativistic guiding center theory[2] modified by the averaging over the two interacting electromagnetic waves.

The theory developed in "I" is suitable for free-electron lasers[3] with a guiding magnetic field in which the gyroperiod is comparable to the period of the signal wave and to the time it takes for the electron beam to travel one wavelength of the wiggler $[\Omega \sim \omega \sim k_{w_\parallel} U_\parallel]$. This is the FEL operating regime which gives maximum efficiency.[4] The guiding center theory of the nonlinear response of a relativistic particle beam in a guide magnetic field to the signal wave and the wiggler field has not been adequately treated in its generality. In this work, we reduce the general relativistic ponderomotive Hamiltonian expression obtained in "I" to free-electron lasers and derive the equations of motion.

We should mention that the Hamiltonian formulation of free-electron lasers has the advantage of expressing the vector evolution equations in terms of a single scalar function on phase space. Furthermore, since the guiding center equations have been averaged over gyrations, they are much more suitable for use in numerical integrations and simulations of free-electron lasers. Namely, instead of integrating along the gyrating trajectories for each of the electrons in the beam, it is possible now, for the averaged equations, to take a two orders of magnitude larger integration steps with equivalent savings in computer time. In addition, because the guiding center transformation has been done in the Hamiltonian, the derived equations conserve phase volume exactly and thus enhancing the equations predictability power. Finally, although we are dealing with single particle motion, the derivation of the averaged Hamiltonian is the major part of the work for a derivation of the nonlinear relativistic gyrokinetic equation for two interacting waves. Once we have the ponderomotive Hamiltonian for the free-electron laser in a guide magnetic field, the Vlasov equation for the distribution of guiding centers F in the Poisson bracket form

$$\frac{\partial F}{\partial t} + \{F, K\} = 0,$$

is the gyrokinetic equation.[5] This will be reported in the future.

In Sec. II, we present a summary of the essential results derived in "I". Because the formulas presented here are so general we can reduce them to arbitrary signal wave polarization and any wiggler field geometry including

guide field nonuniformities. Hence, in Sec. III, as an illustrative application of our theory, we obtain the guiding center relativistic ponderomotive Hamiltonian for free-electron lasers by considering:

(i) the axial guide quasistatic magnetic field to be tapered[6] along its direction of propagation;

(ii) the realizable[7] wiggler field to be tapered[8] both in period and amplitude;

(iii) the signal electromagnetic wave to be a growing modulated wave of arbitrary geometry propagating in the direction of the static guide field.[9]

In Sec. IV., we derive the guiding center free-electron laser equations of motion for the Hamiltonian derived in Sec. III. including the guiding center perpendicular drifts and beam quasistatic self-fields. A conclusion is presented in Sec. V.

II. RELATIVISTIC PONDEROMOTIVE HAMILTONIAN OF TWO INTERACTING WAVES

As derived in "I", the two electromagnetic waves are represented by their scalar and vector potentials ϕ and $\underline{A}$ in an arbitrary gauge. They are assumed to have the form

$$\phi(\underline{x},t) = \sum_{i=1}^{2} \tilde{\phi}_i(\underline{x},t)e^{i\psi_i(\underline{x},t)} + c.c., \tag{1a}$$

and

$$\underline{A}(\underline{x},t) = \sum_{i=1}^{2} \tilde{\underline{A}}_i(\underline{x},t)e^{i\psi_i(\underline{x},t)} + c.c, \tag{1b}$$

where the over-tilde represents the slowly varying amplitude of the wave packet. The local wavenumbers k_1 and k_2 and frequencies ω_1 and ω_2 of the waves are given by

$$\underline{k}_i(\underline{x},t) = \nabla\psi_i(\underline{x},t), \quad i = 1,2, \tag{2a}$$

and

$$\omega_i(\underline{x},t) = -\frac{\partial\psi_i(\underline{x},t)}{\partial t}, \; i = 1,2. \tag{2b}$$

The relativistic particle Hamiltonian is given by

$$H(\underset{\sim}{x},\underset{\sim}{p}) = \{m^2c^4 + c^2[\underset{\sim}{p} - \frac{e}{c}\underset{\sim}{A}_0(\underset{\sim}{x},t) - \frac{e}{c}\underset{\sim}{A}(\underset{\sim}{x},t)]\}^{1/2}$$

$$+ e\phi_0(\underset{\sim}{x},t) + e\phi(\underset{\sim}{x},t), \qquad (3)$$

where we allow the unperturbed fields, as given by ϕ_0 and $\underset{\sim}{A}_0$, to be slow functions of position and time. Physically, this means that the scale length of the spatial inhomogeneities in ϕ_0 and $\underset{\sim}{A}_0$ (or $\underset{\sim}{E}_0$ and $\underset{\sim}{B}_0$) are smaller than the gyroradius, which is the standard approximation in guiding center theory and true in free-electron lasers. In addition, it means that we allow ϕ_0 and $\underset{\sim}{A}_0$ to change appreciably on a drift time scale.

Because we are interested in the ponderomotive Hamiltonian, which is a second order effect, we expand (3) up to second order in the waves amplitudes. The result is

$$H = H_0 + H_1 + H_2,$$

where

$$H_0 = e\phi_0 + mc^2\gamma_0, \qquad (4a)$$

$$H_1 = e\phi - \frac{e}{\gamma_0 c}\underset{\sim}{u}_0 \cdot \underset{\sim}{A}, \qquad (4b)$$

and

$$H_2 = \frac{e^2}{2\gamma_0 mc^2}[\underset{\sim}{A}^2 - \frac{1}{\gamma_0^2 c^2}(u_0 \cdot \underset{\sim}{A})^2]. \qquad (4c)$$

As discussed in Ref. 2, we introduced a velocity-like variable $\underline{u}_0$ by

$$m\underline{u}_0 = \underline{p} - \frac{e}{c}\underline{A}_0 = m\underline{u} + \frac{e}{c}\underline{A},$$

where $\underline{u} = \gamma\underline{v}$ is the world velocity and

$$\gamma_0 = \left(1 + \frac{u_0^2}{c^2}\right)^{1/2}.$$

$\gamma_0 mc^2$ is the energy of the injected relativistic beam. The introduction of $\underline{u}_0$ allows us to exhibit the perturbation in the Hamiltonian explicitly instead of in the Poisson brackets.

Next we transform the Hamiltonian (4) to the relativistic guiding center variables, by the transformation $(\underline{x},\underline{u}_0) \to (\underline{X},U_\parallel,\mu,\Theta)$. The first three variables $\underline{X}$ represent the location, in physical space, of the particle's guiding center. The variable $U_\parallel$ is called the parallel world velocity of the guiding center; it is, except for the relativistic factor, the parallel guiding center drift to lowest order in the waves fields. The next variable μ is the magnetic moment and is related to the perpendicular guiding center drift by $\mu = mU_\perp^2/2B_0$, where $U_\perp$ is essentially the gyrophase average of $u_{0_\perp}$. Finally, Θ is the gyroaveraged gyrophase whose time-derivative to lowest order is the local relativistic gyrofrequency. The new guiding center variables are defined by[2]

$$\underline{X} = \underline{x} - \varepsilon\frac{mcu_{0_\perp}}{eB_0}\hat{a} + O(\varepsilon^2), \tag{5a}$$

$$U_\parallel = u_{0_\parallel} + O(\varepsilon), \tag{5b}$$

$$\mu = \frac{m u_{0_\perp}^2}{2B_0} + O(\varepsilon), \tag{5c}$$

and

$$\Theta = \theta + O(\varepsilon), \tag{5d}$$

where the fields are evaluated at $(\underline{x},t)$ and ε is the dimensionless parameter which indicates the order of various terms in the guiding center expansion. Since in this section we are presenting only results, we will set $\varepsilon = 1$ from now on to recover the physical formulas. The instantaneous gyrophase θ is defined implicitly by

$$\hat{a} = \cos\theta\hat{\tau}_1 - \sin\theta\hat{\tau}_2, \tag{6a}$$

and

$$\hat{c} = -\sin\theta\hat{\tau}_1 - \cos\theta\hat{\tau}_2. \tag{6b}$$

The perpendicular unit vectors $\hat{a}$ and $\hat{c}$ rotate with the particle. $\hat{a}$ is in the direction of the gyroadius as given by (5a) and $\hat{c}$ is given by $\underline{u}_0 = u_{0_\parallel}\hat{b} + u_{0_\perp}\hat{c}$, where $\hat{b} = \underline{B}_0/B_0$. The triad $(\hat{a},\hat{b},\hat{c})$ obeys $\hat{a} = \hat{b} \times \hat{c}$. The perpendicular unit vectors do not rotate with the particle and satisfy $\hat{\tau}_1 \times \hat{\tau}_2 = \hat{b}$. They have a slow dependence on $(\underline{x},t)$ since $\underline{B}_0$ itself does.

For the Hamiltonian structure to be complete we need the Poisson brackets among the relativistic guiding center variables. The guiding center variables are noncanonical and the nonvanishing Poisson brackets are given by[2]

$$\{X_i, X_j\} = -\frac{c}{e}\frac{b_{ij}}{B^*_{0_\parallel}}, \tag{7a}$$

$$\{\underline{X}, U_\parallel\} = \frac{1}{mB^*_{0_\parallel}} \underline{B}^*_0, \tag{7b}$$

and

$$\{\Theta, \mu\} = \frac{e}{mc}, \tag{7c}$$

where b_{ij} is the tensor dual to the unit vector $\hat{b}$ $[b_{ij}\ T_j = -(\hat{b} \times \underline{T})_i$ for any $\underline{T}]$ and B^*_0 is defined by

$$\underline{B}^*_0 = \underline{B}_0 + \frac{mc}{e} U_\parallel \nabla \times \hat{b}(\underline{X}); \tag{8}$$

and $\nabla \equiv \partial/\partial\underline{X}$. The Poisson brackets are exact to all orders in ε. In terms of the fundamental Poisson brackets (7), the Poisson bracket of two arbitrary phase functions f and g is given by the chain rule[2] :

$$\{f,g\} = -\frac{c}{eB^*_{0_\parallel}} \hat{b} \cdot (\nabla f \times \nabla g) + \frac{1}{mB^*_{0_\parallel}} \underline{B}_0 \cdot \left(\nabla f \frac{\partial g}{\partial U_\parallel} - \nabla g \frac{\partial f}{\partial U_\parallel}\right)$$

$$+ \frac{e}{mc}\left(\frac{\partial f}{\partial \Theta}\frac{\partial g}{\partial \mu} - \frac{\partial f}{\partial \mu}\frac{\partial g}{\partial \Theta}\right). \tag{9}$$

This formula will be useful in deriving the equations of motion from the guiding center Hamiltonian.

In terms of the guiding center variables (5), the unperturbed Hamiltonian (4a) becomes

$$H_0 = e\phi_0 + mc^2\Gamma, \tag{10}$$

where

$$\Gamma = \left(1 + \frac{2\mu B_0(\underset{\sim}{X},t)}{mc^2} + \frac{U_\parallel^2}{c^2}\right)^{1/2}. \tag{11}$$

The first order Hamiltonian (4b) is given by the following Fourier expansion[1]

$$H_1 = \sum_{i=1}^{2} \sum_{\ell=-\infty}^{+\infty} \{H_{i_\ell}\, e^{i\psi_i(X,t)+i\ell[\Theta + \alpha_i(\underset{\sim}{X},t) + \frac{\pi}{2}]} + \text{c.c.}\}, \tag{12}$$

where H_{i_ℓ} is the ℓth Θ-Fourier coefficient of the first order perturbation due to the ith wave and $\alpha_i(\underset{\sim}{X},t)$ is the angle between $\underset{\sim}{k}_{i_\perp}$ and $\hat{\tau}_1$. We introduce the triads $(\hat{k}_{i_\perp}, \hat{b} \times k_{i_\perp}, \hat{b})$ for i = 1, 2, in which $\hat{k}_{i_\perp} = \cos\alpha_i\, \hat{\tau}_1 + \sin\alpha_i\, \hat{\tau}_2$. The Fourier coefficients are then given by

$$H_{i_\ell} = eJ_\ell^i\, \tilde{\phi}_i - \frac{e}{c}\, \underset{\sim}{\tilde{A}}_i \cdot [V_\parallel J_\ell^i \hat{b} + \frac{\Omega}{k_{i_\perp}} (\ell J_\ell^i \hat{k}_{i_\perp} + 2i\mu \frac{\partial J_\ell^i}{\partial \mu}\, \hat{b} \times \hat{k}_{i_\perp})], \tag{13}$$

where $J_\ell^i \equiv J_\ell(k_{i_\perp}\rho)$, $\rho = \left(\frac{2mc^2\mu}{e^2B_0}\right)^{1/2}$, $\Omega = \frac{eB_0}{\Gamma mc}$, and $V_\parallel = U_\parallel/\Gamma$. As for the second order Hamiltonian, we only keep the slow terms in Eq. (4c) since we are interested in the ponderomotive Hamiltonian. We average over gyration and oscillation of both waves but keep the slow beating terms of the form $\exp i[\psi_1(\underset{\sim}{X},t) - \psi_2(\underset{\sim}{X},t)]$. The result is[1]

$$\begin{aligned}
\bar{H}_2 = \frac{e^2}{\Gamma^3 mc^2} \Big(&|\underset{\sim}{\tilde{A}}_1|^2+|\underset{\sim}{\tilde{A}}_2|^2 + J_0(\Delta k_\perp\rho)\, [\underset{\sim}{\tilde{A}}_1 \cdot \underset{\sim}{\tilde{A}}_2^* e^{i(\psi_1 - \psi_2)} + \text{c.c.}] \\
&+ \frac{1}{mc^2} \{(\mu B_0 + mU_\parallel^2)\, [|\underset{\sim}{\tilde{A}}_{1_\perp}|^2+|\underset{\sim}{\tilde{A}}_{2_\perp}|^2 + J_0(\Delta k_\perp\rho)(\underset{\sim}{\tilde{A}}_{1_\perp} \cdot \underset{\sim}{\tilde{A}}_{2_\perp}^*\, e^{i(\psi_1 - \psi_2)} \\
&+ \text{c.c.}) + 2\mu B_0\, [|\tilde{A}_{1_\parallel}|^2+|\tilde{A}_{2_\parallel}|^2+ J_0(\Delta k_\perp\rho)(\tilde{A}_{1_\parallel} \tilde{A}_{2_\parallel}^* e^{i(\psi_1-\psi_2)} + \text{c.c.})]\}\Big),
\end{aligned} \tag{14}$$

where $\Delta k_\perp = [k_{1_\perp}^2 + k_{2_\perp}^2 - 2k_{1_\perp} k_{2_\perp} \cos(\alpha_1 - \alpha_2)]^{1/2}$ and we made use of the Bessel identity

$$\sum_{\ell=-\infty}^{+\infty} e^{i\ell(\alpha_1 - \alpha_2)} J_\ell(k_1\rho) J_\ell(k_2\rho) = J_0(\Delta k_\perp \rho).$$

At this point we subject the Hamiltonian $H = H_0 + H_1 + \bar{H}_2$ to an averaging transformation to eliminate gyration and fast oscillations contained in H_1 but keeping the beating terms $\exp i(\psi_1 - \psi_2)$. This canonical transformation is effected by using Lie transforms which leads to the following Θ-independent Hamiltonian

$$K = K_0 + K_2, \tag{15}$$

where $K_0 = H_0$ as given by Eq. (10), or

$$K_0 = e\phi_0 + mc^2\Gamma, \tag{16}$$

and the ponderomotive Hamiltonian K_2 contains two parts

$$K_2 = K_2^a + K_2^b. \tag{17}$$

The first part $K_2^a = \bar{H}_2$ which is given by Eq. (14). The second part comes from the canonical transformation of H_1 and can be expressed as[1]

$$K_2^b = \sum_{i=1}^{2} \sum_{\ell=-\infty}^{+\infty} \left(\frac{e\ell}{mc}\frac{\partial}{\partial\mu} + \frac{k_{i_\parallel}}{m}\frac{\partial}{\partial U_\parallel}\right) \frac{|H_{i_\ell}|^2 + \frac{1}{2}[H_{1_\ell} H_{2_\ell}^* e^{i(\psi_1-\psi_2) + i\ell(\alpha_1-\alpha_2)} + \text{c.c.}]}{\omega_i - k_{i_\parallel} V_\parallel - \ell\Omega}, \tag{18}$$

Where H_{1_ℓ} and H_{2_ℓ} are given by Eq. (13). It is possible to absorb $\exp i\ell(\alpha_1 - \alpha_2)$ into $H^*_{2_\ell}$. In this case the triad of unit vectors for both waves is $(\hat{k}_{1_\perp}, \hat{b} \times \hat{k}_{1_\perp}, \hat{b})$ in which we drop the subscript 1.

III. RELATIVISTIC PONDEROMOTIVE HAMILTONIAN FOR FREE-ELECTRON LASERS

We consider a relativistic electron beam propagating in the tapered guide field $\underline{B}_0 = B_0(Z,t)\hat{b}$. The free-electron laser effect results from the bunching of the beam due to the beating of the signal wave, represented by the potentials ϕ_s and $\underline{A}_s$, and the magnetostatic wiggler $\underline{A}_w$. For the purpose of illustrating our theory, we reduce the relativistic guiding center Hamiltonian of two interacting waves, as given by Eqs. (13) - (18), by assuming the following signal and wiggler fields geometry:

(1) The signal electromagnetic wave is a growing modulated wave of arbitrary geometry propagating in the Z direction.

$$\underline{A}_s(\underline{X},t) = \sum_n \tilde{\underline{A}}_{s_\perp}(\underline{X},t) e^{i\int_0^Z [k_{s_\parallel}(Z') + nk_{w_\parallel}(Z')]dZ' - i\omega t} + \text{c.c.} \tag{19}$$

and

$$\phi_s(\underline{X},t) = \sum_n \tilde{\phi}_s(\underline{X},t) e^{i\int_0^Z [k_{s_\parallel}(Z') + nk_{w_\parallel}(Z')]dZ' - i\omega t} + \text{c.c.}, \tag{20}$$

where the sum is over the wiggler harmonics and the subscripts $\perp$ and $\parallel$ indicates perpendicular and parallel to the local $\hat{b}$, respectively.

(2) The realizable wiggler field is tapered both in period and amplitude

$$\underset{\sim}{A}_w(\underset{\sim}{X},t) = \underset{\sim}{\tilde{A}}_{w_\perp}(\underset{\sim}{X},t)e^{i\int_0^Z k_{w_\parallel}(Z')dZ'} + c.c.. \tag{21}$$

As indicated by Eq. (17), we split the ponderomotive Hamiltonian in two parts. The first part K_2^a is given by Eq. (14) and its reduction for free electron lasers is straightforward by identifying, say, $\underset{\sim}{A}_1 = \underset{\sim}{A}_s$ and $\underset{\sim}{A}_2 = \underset{\sim}{A}_w$. Then for the signal and wiggler fields, as given by Eqs. (19) and (21), $J_0(\Delta k_\perp \rho) = 1$ and we obtain

$$K_2^a = \frac{e^2}{\Gamma mc^2}\left(1 - \frac{\mu B_0}{\Gamma^2 mc^2}\right)\left[\,|\underset{\sim}{\tilde{A}}_{s_\perp}|^2 + |\underset{\sim}{\tilde{A}}_{w_\perp}|^2\right.$$

$$\left. + 2\sum_n |\underset{\sim}{\tilde{A}}_{s_\perp} \cdot \underset{\sim}{\tilde{A}}^*_{w_\perp}| \cos\left(\int_0^Z [k'_\parallel(Z') - k_{w_\parallel}(Z')]dZ' - \omega t\right)\right], \tag{22}$$

where Γ is given by Eq. (11) and $k'_\parallel = k_{s_\parallel} + nk_{w_\parallel}$. We observe that the fields amplitudes $|\underset{\sim}{\tilde{A}}_{s_\perp}|$ and $|\underset{\sim}{\tilde{A}}_{w_\perp}|$ depend on $(\underset{\sim}{X},t)$.

The second part K_2^b is given by Eq. (18) where H_{s_ℓ} and H_{w_ℓ} is given by Eq. (13). The reduction of K_2^b for free electron laser fields (19) - (21) is more involved than for K_2^a. For both signal and wiggler, only the $\ell = 0$ and $\ell = \pm 1$ Fourier components are different from zero, as given by

$$H_s\Big|_{\ell = 0} = e\tilde{\phi}_s, \tag{23a}$$

$$H_w\Big|_{\ell = 0} = 0, \tag{23b}$$

$$H_s\Big|_{\ell = \pm 1} = -\frac{1}{2}\frac{e}{c}\Omega\rho\tilde{A}_{s_x} \mp \frac{i\mu}{\Gamma\rho}\tilde{A}_{s_y}, \tag{23c}$$

and

$$H_w\Big|_{\ell = \pm 1} = -\frac{1}{2}\frac{e}{c}\Omega\rho\tilde{A}_{w_x} \mp \frac{i\Omega}{\Gamma\rho}\tilde{A}_{w_y}, \tag{23d}$$

where we absorbed $\exp i\ell(\alpha_s - \alpha_w)$ in $H^*_{w_\ell}$, e is the signed electron charge, and we chose $\hat{x} = \hat{k}_\perp$ and $\hat{y} = \hat{b} \times \hat{k}_\perp$. Therefore, A_{s_x}, A_{s_y} and A_{w_x}, A_{w_y} are in relation to the x,y frame. In view of expressions (23), K_2^b can be written as the Cartesian components of the vector potentials

$$K_2^b = K_2^b\Big|_{\ell = 0} + K_2^b\Big|_{\ell = 1} + K_2^b\Big|_{\ell = -1}, \tag{24}$$

which are obtained from Eq. (18). For $\ell = 0$, we obtain

$$K_2^b\Big|_{\ell = 0} = \sum_n \frac{k_\parallel'}{m}\frac{\partial}{\partial U_\parallel}\left(\frac{|e\tilde{\phi}_s|^2}{\omega - k_\parallel' V_\parallel}\right). \tag{25}$$

In taking the derivative in Eq. (25) we should keep in mind that $V_\parallel = U_\parallel/\Gamma$ where Γ depends on $U_\parallel$ as given by Eq. (11). Thus, we obtain

$$K_2^b\Big|_{\ell = 0} = \sum_n \frac{e^2 k_\parallel'^2 |\tilde{\phi}_s|^2 \left(1 + \frac{U_\perp^2}{c^2}\right)}{\Gamma^3 m(\omega - k_\parallel' V_\parallel)^2}. \tag{26}$$

To evaluate $K_2^b\Big|_{\ell = \pm 1}$, we need the absolute values of Eqs. (23c) and (23d) and their cross product as given by

$$\left|H_s\Big|_{\ell = \pm 1}\right|^2 = \frac{1}{4}\frac{e^2 V_\perp^2}{c^2}\left|\tilde{\underline{A}}_{s_\perp}\right|^2, \tag{27a}$$

$$\left|H_w\Big|_{\ell = \pm 1}\right|^2 = \frac{1}{4}\frac{e^2 V_\perp^2}{c^2}\left|\tilde{\underline{A}}_{w_\perp}\right|^2, \tag{27b}$$

and

$$H_s\Big|_{\ell=\pm 1} H_w^*\Big|_{\ell=\pm 1} = \frac{1}{4}\frac{e^2V_\perp^2}{c^2}(\tilde{\underline{A}}_{s_\perp} \cdot \tilde{\underline{A}}_{w_\perp}^* \mp i\tilde{\underline{A}}_{s_\perp} \times \tilde{\underline{A}}_{w_\perp}^*), \qquad (27c)$$

where $\rho = V_\perp/\Omega$. Hence, substituting expressions (27) in Eq. (18) we obtain

$$K_2\Big|_{\ell=\pm 1} = \sum_n [(\frac{k_\parallel'}{m}\frac{\partial}{\partial U_\parallel} \pm \frac{e}{mc}\frac{\partial}{\partial \mu})\frac{\frac{1}{4}\frac{e^2V_\perp^2}{c^2}R_s^\pm}{\omega - k_\parallel' V_\parallel \mp \Omega}$$

$$+ (\frac{k_{w_\parallel}}{m}\frac{\partial}{\partial U_\parallel} \pm \frac{e}{mc}\frac{\partial}{\partial \mu})[\frac{\frac{1}{4}\frac{e^2V_\perp^2}{c^2}R_w^\pm}{-k_{w_\parallel} V_\parallel \mp \Omega}], \qquad (28)$$

where

$$R_s^\pm \equiv |\tilde{\underline{A}}_{s_\perp}|^2 + |\tilde{\underline{A}}_{s_\perp} \cdot \tilde{\underline{A}}_{w_\perp}^*|\cos(\int_0^Z [k_\parallel'(Z') - k_{w_\parallel}(Z')]\,dZ' - \omega t)$$

$$\pm |\tilde{\underline{A}}_{s_\perp} \times \tilde{\underline{A}}_{w_\perp}^*|\sin(\int_0^Z [k_\parallel'(Z') - k_{w_\parallel}(Z')]dZ' - \omega t), \qquad (29a)$$

and

$$R_w^\pm \equiv |\tilde{\underline{A}}_{w_\perp}|^2 + |\tilde{\underline{A}}_{s_\perp} \cdot \tilde{\underline{A}}_{w_\perp}^*|\cos(\int_0^Z [k_\parallel'(Z') - k_{w_\parallel}(Z')]\,dZ' - \omega t)$$

$$\pm |\tilde{\underline{A}}_{s_\perp} \times \tilde{\underline{A}}_{w_\perp}^*|\sin(\int_0^Z [k_\parallel'(Z') - k_{w_\parallel}(Z')]\,dZ' - \omega t). \qquad (29b)$$

To evaluate the derivatives in Eq. (28) we should again realize that $V_\perp$ and Ω depend on Γ which in turn depends on μ and $U_\parallel$. Taking that into account, we obtain

$$K_2\Big|_{\ell = \pm 1} = \sum_n \frac{e^2 V_\perp^2}{4\Gamma mc^4} \{[1 + \frac{k_{w_\parallel}^2 c^2}{(-k_{w_\parallel} V_\parallel \mp \Omega)^2}] R_w^\pm$$

$$+ [2 \frac{-k_\parallel' V_\parallel \mp \Omega}{\omega - k_\parallel' V_\parallel \mp \Omega} + \frac{k_\parallel'^2 c^2 - (-k_\parallel' V_\parallel \mp \Omega)^2}{(\omega - k_\parallel' V_\parallel \mp \Omega)^2}] R_s^\pm\}. \tag{30}$$

Substituting Eqs. (26) and (30) into Eq. (24), we obtain the expression for the second part of the ponderomotive Hamiltonian which is given by

$$K_2^b = \sum_n \frac{e^2 k_\parallel'^2 |\tilde{\phi}_s|^2 (1 + \frac{2\mu B_0}{mc^2})}{\Gamma^3 m(\omega - k_\parallel' V_\parallel)^2}$$

$$+ \sum_n \frac{e^2 V_\perp^2}{4\Gamma mc^4} \{[1 + \frac{k_{w_\parallel}^2 c^2}{(-k_{w_\parallel} V_\parallel - \Omega)^2}] R_w^+$$

$$+ [1 + \frac{k_{w_\parallel}^2 c^2}{(-k_{w_\parallel} V_\parallel + \Omega)^2}] R_w^- + [2 \frac{-k_\parallel' V_\parallel - \Omega}{\omega - k_\parallel' V_\parallel - \Omega} + \frac{k_\parallel'^2 c^2 - (-k_\parallel' V_\parallel - \Omega)^2}{(\omega - k_\parallel' V_\parallel - \Omega)^2}] R_s^+$$

$$+ [2 \frac{-k_\parallel' V_\parallel + \Omega}{\omega - k_\parallel' V_\parallel + \Omega} + \frac{k_\parallel'^2 c^2 - (-k_\parallel' V_\parallel + \Omega)^2}{(\omega - k_\parallel' V_\parallel + \Omega)^2}] R_s^-\}. \tag{31}$$

IV. FREE-ELECTRON LASER EQUATIONS OF MOTION

The equations of motion for the relativistic guiding center Hamiltonian resulting from the beating of the wiggler and signal fields are found with the help of Eq. (9). We obtain six equations of motion corresponding to each one of the dynamical variables.

A. Guiding center magnetic moment.

$$\dot{\mu} = \{\mu, K\} = 0, \text{ or, } \mu = \text{constant.} \tag{32}$$

Eq. (32) represents the adiabatic invariance of the magnetic moment; it is not an exact invariant because the ponderomotive Hamiltonian still depends on the gyrophase to order cubic in the fields. As it was shown in "I", the adiabatic magnetic moment can be expressed in terms of $mU_\perp^2/2B_0$, which contains the fast oscillations of the signal and wiggler, by

$$\mu = \frac{mU_\perp^2}{2B_0} - \frac{e}{mc}\left\{\sum_{\ell = \pm 1}\left[\frac{\ell\, \tilde{H}_{w_\ell} \exp(i\Phi_{w_\ell})}{-k_{w_\parallel} V_\parallel - \ell\Omega} + \sum_n \frac{\ell\, \tilde{H}_{s_\ell} \exp(i\Phi_{s_\ell})}{\omega - k_\parallel V_\parallel - \ell\Omega}\right] + \text{c.c.}\right\}, \tag{33}$$

where

$$\Phi_{w_\ell} = \psi_w(\underline{X}, t) + \ell\,[\Theta + \alpha_w(\underline{X}, t) + \pi/2],$$

and similar expression for Φ_{s_ℓ}.

B. Nonlinear gyrofrequency.

$$\dot{\Theta} = \{\Theta, K\} = \Omega + \frac{e}{mc}\frac{\partial K_2}{\partial \mu}.$$

The last term is the nonlinear gyrofrequency shift due to the ponderomotive potential. The μ derivative of Eq. (22) gives

$$\frac{e}{mc}\frac{\partial K_2^a}{\partial \mu} = -\sum_n \frac{2e^2\Omega}{\Gamma^2 m^2 c^4}\left(1 - \frac{3}{2}\frac{V_\perp^2}{c^2}\right)\left\{\left|\tilde{\underline{A}}_{s_\perp}\right|^2 + \left|\tilde{\underline{A}}_{w_\perp}\right|^2 + 2\left|\tilde{\underline{A}}_{s_\perp}\cdot\tilde{\underline{A}}_{w_\perp}\right|\cos\left[\int_0^Z (k_\parallel' - k_{w_\parallel})dZ' - \omega t\right]\right\}. \tag{35}$$

We use Eq. (31) to evaluate $\partial K_2^b/\partial\mu$, which gives

$$\frac{e}{mc}\frac{\partial K_2^b}{\partial \mu} = \sum_n \frac{2e^2\Omega k_\parallel'^2|\tilde{\phi}_s|^2}{\Gamma^2 m^2 c^2(\omega - k_\parallel' V_\parallel)^2}\left[1 - \frac{1}{\Gamma^2}\left(1 + \frac{U_\perp^2}{c^2}\right)\left(\frac{3}{2} + \frac{k_\parallel' V_\parallel}{\omega - k_\parallel' V_\parallel}\right)\right]$$

$$+ \sum_n \frac{e^2\Omega}{2\Gamma^2 m^2 c^4}\left\{\left[1 + \frac{k_{w_\parallel}^2 c^2}{(-k_{w_\parallel} V_\parallel - \Omega)^2}\right] R_w^+ + \left[1 + \frac{k_{w_\parallel}^2 c^2}{(-k_{w_\parallel} V_\parallel - \Omega)^2}\right] R_w^-\right.$$

$$\left. + \left(\frac{2r^+}{\omega_+} + \frac{k_\parallel'^2 c^2 - r_+^2}{\omega_+^2}\right) R_s^+ + \left(\frac{2r^-}{\omega_-} + \frac{k_\parallel'^2 c^2 - r_-^2}{\omega_-^2}\right) R_s^-\right\}$$

$$- \sum_n \frac{e^2\Omega V_\perp^2}{4\Gamma^2 m^2 c^6}\left\{\left[3 + \frac{k_{w_\parallel}^2 c^2}{(-k_{w_\parallel} V_\parallel - \Omega)^2}\right] R_w^+ + \left[3 + \frac{k_{w_\parallel}^2 c^2}{(-k_{w_\parallel} V_\parallel + \Omega)^2}\right] R_w^-\right.$$

$$+ \left[\frac{8r_+}{\omega_+} - \frac{7r_+^2}{\omega_+^2} + \frac{2r_+^3}{\omega_+^3} + \frac{k_\parallel' c^2}{\omega_+^2}\left(3 - \frac{2r_+}{\omega_+}\right)\right] R_s^+$$

$$+ \left[\frac{8r_-}{\omega_-} - \frac{7r_-^2}{\omega_-^2} + \frac{2r_-^3}{\omega_-^3} + \frac{k_\parallel'^2 c^2}{\omega_-^2}\left(3 - \frac{2r_-}{\omega_-}\right)\right]\Big\} R_s^-, \quad (36)$$

$$r_\pm \equiv - k_\parallel' V_\parallel \mp \Omega \text{ and } \omega_\pm \equiv \omega - k_\parallel' V_\parallel \mp \Omega. \quad (37)$$

The sum of Eqs. (35) and (36) is the nonlinear gyrofrequency shift indicated in Eq. (34).

C. Parallel force on a guiding center.

$$m\dot{U}_\parallel = \{mU_\parallel, K_0 + K_2\} = - \hat{b} \cdot \nabla(K_0 + K_2). \quad (38)$$

From Eq. (10) we obtain

$$- \hat{b} \cdot \nabla K_0 = \hat{b} \cdot \left(e\underline{E}_0 - \frac{\mu}{\Gamma} \nabla B_0\right), \quad (39)$$

where $\underline{E}_0(\underline{X},t)$ is the electric field due to the beam self-fields and $B_0(Z,t)$ is the tapered guide field. From Eqs. (22) and (31) we obtain the contribution of the ponderomotive Hamiltonian to the parallel force, which is given by

$$-\hat{b} \cdot \nabla K_2 = \sum_n \frac{2e^2}{\Gamma mc^2}\left(1 - \frac{\mu B_0}{\Gamma^2 mc^2}\right)(k_\parallel' - k_{w_\parallel})\left|\underline{\tilde{A}}_{s_\perp} \cdot \underline{\tilde{A}}_{w_\perp}\right| \sin\left[\int^Z (k_\parallel' - k_{w_\parallel}) dZ' - \omega t\right]$$

$$+ \frac{e^2 V_\perp^2 (k_\parallel' - k_{w_\parallel})}{4\Gamma mc^4} \{T_1 |\tilde{\underline{A}}_{s_\perp} \cdot \tilde{\underline{A}}^*_{w_\perp}| \sin[\int_0^Z (k'_\parallel - k_{w_\parallel}) dZ' - \omega t]$$

$$- T_2 |\tilde{\underline{A}}_{s_\perp} \times \tilde{\underline{A}}^*_{w_\perp}| \cos[\int_0^Z (k'_\parallel - k_{w_\parallel}) dZ' - \omega t]\}, \tag{40}$$

where

$$T_1 = 2 + \frac{k_{w_\parallel}^2 c^2}{(-k_{w_\parallel} V_\parallel - \Omega)^2} + \frac{k_{w_\parallel}^2 c^2}{(-k_{w_\parallel} V_\parallel + \Omega)^2} + \frac{2r_+}{\omega_+} + \frac{2r_-}{\omega_-}$$

$$+ \frac{k_\parallel'^2 c^2 - r_+^2}{\omega_+^2} + \frac{k_\parallel'^2 c^2 - r_-^2}{\omega_-^2}, \tag{41}$$

and

$$T_2 = \frac{k_{w_\parallel}^2 c^2}{(-k_{w_\parallel} V_\parallel - \Omega)^2} - \frac{k_{w_\parallel}^2 c^2}{(-k_{w_\parallel} + \Omega)^2} + \frac{2r_+}{\omega_+} - \frac{2r_-}{\omega_-}$$

$$+ \frac{k_\parallel'^2 c^2 - r_+^2}{\omega_+^2} - \frac{k_\parallel'^2 c^2 - r_-^2}{\omega_-^2}. \tag{42}$$

D. Guiding center drifts

$$\dot{\underline{X}} = \frac{\hat{b}}{B_0} \times \left(\frac{c}{\Gamma e} \mu \nabla B_0 - c\underline{E}_0\right) + \hat{b} \left(V_\parallel + \frac{1}{m} \frac{\partial K_2}{\partial U_\parallel}\right). \tag{43}$$

The first term in Eq. (43) are the perpendicular drifts. The second term is the guiding center parallel velocity which differs from $V_\parallel$ by the ponderomotive term correction. Explicitly, $(1/m)\ \partial K_2/\partial U_\parallel$ is evaluated using expressions (22) and (31) which yields

$$\frac{1}{m}\frac{\partial K_2^a}{\partial U_\parallel} = -\sum_n \frac{e^2 V_\parallel}{\Gamma^2 mc^4}\left(1 - \frac{3\mu B_0}{\Gamma^2 mc^2}\right)\left\{\left|\tilde{\underline{A}}_{s_\perp}\right|^2 + \left|\tilde{\underline{A}}_{w_\perp}\right|^2 + 2\left|\tilde{\underline{A}}_{s_\perp}\cdot\tilde{\underline{A}}^*_{w_\perp}\right|\cos\left[\int_0^Z (k'_\parallel - k_w)dZ' - \omega t\right]\right\} \tag{44}$$

and

$$\frac{1}{m}\frac{\partial K_2^b}{\partial U_\parallel} = -\sum_n \frac{e^2 k_\parallel'^2 |\tilde{\phi}_s|^2}{m^2\Gamma^4(\omega - k'_\parallel V_\parallel)^2}\left(1 + \frac{U_\perp^2}{c^2}\right)\left[\frac{3V_\parallel}{c^2} - \frac{2k'_\parallel}{\Gamma^2(\omega - k'_\parallel V_\parallel)}\right],$$

$$-\sum_n \frac{3e^2V_\perp^2 V_\parallel}{4\Gamma^2 m^2 c^6}\left\{\left[1 + \frac{k_{w_\parallel}^2 c^2}{(-k_{w_\parallel}V_\parallel - \Omega)^2}\right] R_w^+ + \left[1 + \frac{k_{w_\parallel}^2 c^2}{(-k_{w_\parallel}V_\parallel + \Omega)^2}\right] R_w^-,\right.$$

$$\left. + \left(\frac{2r_+}{\omega_+} + \frac{k_\parallel'^2 c^2 - r_+^2}{\omega_+^2}\right) R_s^+ + \left(\frac{2r_-}{\omega_-} + \frac{k_\parallel'^2 c^2 - r_-^2}{\omega_-^2}\right) R_s^-\right\},$$

$$+\sum_n \frac{e^2V_\perp^2}{4\Gamma^2 m^2 c^4 V_\parallel}\left\{\frac{2k_{w_\parallel}^2 c^2}{(-k_{w_\parallel}V_\parallel - \Omega)^2}\left(\frac{k_{w_\parallel}V_\parallel}{-k_{w_\parallel}V_\parallel - \Omega} + \frac{V_\parallel^2}{c^2}\right) R_w^+,\right.$$

$$+ \frac{2k_{w_\parallel}^2 c^2}{(-k_{w_\parallel}V_\parallel + \Omega)^2}\left(\frac{k_{w_\parallel}V_\parallel}{-k_{w_\parallel}V_\parallel + \Omega} + \frac{V_\parallel^2}{c^2}\right) R_w^-,$$

$$+ \frac{2\omega^2(-k_\parallel' V_\parallel - r_+ \frac{V^2}{c^2})}{\omega_+^3} \left(1 - \frac{k_\parallel^2 c^2}{\omega^2}\right) R_s^+ ,$$

$$+ \frac{2\omega^2(-k_\parallel' V_\parallel - r_- \frac{V_\parallel^2}{c^2})}{\omega_-^3} \left(1 - \frac{k_\parallel^2 c^2}{\omega^2}\right) R_s^- \} . \tag{45}$$

The sum of Eqs. (44) and (45) are the ponderomotive correction to the parallel velocity drift.

Equations (32), (34), (38), and (43) are the complete guiding center equations for free electron lasers. Having these equations at hand, we can obtain the equation for the energy variation of the particles which is given by[1,2]

$$\frac{dE}{dt} = \frac{\partial(K_0 + K_2)}{\partial t} - \frac{e}{c} \dot{\underline{X}} \cdot \frac{\partial \underline{A}^*_{TOTAL}}{\partial t} , \tag{46}$$

where $\underline{A}^*_{TOTAL} = \underline{A}_0 + \underline{A}_s + \underline{A}_w + \frac{mc}{e} U_\parallel \hat{b}$ and $\dot{\underline{X}}$ is given by Eq. (43). Similarly, we can straightforwardly write the equation for the phase variation, which is given by

$$\frac{d\Phi}{dt} = - \omega + [k_\parallel'(X_\parallel) - k_{w_\parallel}(X_\parallel)] \dot{X}_\parallel , \tag{47}$$

where $\dot{X}_\parallel$ is obtained from Eq. (43).

V. CONCLUSIONS

We have obtained a general expression for the ponderomotive Hamiltonian for free-electron lasers as given by Eqs. (22) and (31). From this expression we derived the general guiding center equations of motion governing the full free-electron laser phase space dynamics. They can be simplified depending on the specific experimental system under consideration. For instance, whole terms will drop out just by assuming $U_{\perp}^2 \ll U_{\parallel}^2$. Further simplifications can be done by keeping the terms with the right resonant denominator for a given problem.

We should also mention that although we assume $\underline{X}$ to be Cartesian in our derivation, we can express $\underline{X}$ in terms of other systems of coordinates, say, cylindrical, flux, or Hamada, by the application of the chain rule.

Our final expressions are given in terms of the scalar potential $\tilde{\phi}(\underline{X},t)$ and the perpendicular vector potentials $\underline{\tilde{A}}_{s_{\perp}}(\underline{X},t)$ and $\underline{\tilde{A}}_{w_{\perp}}(\underline{X},t)$. It is necessary to choose the appropriate form of these realizable potentials for a given experimental situation.

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